CS 638
Homework 1: Linear Algebra and Geometry Review

Due: Thursday, September 9, 1999 9:30AM (the beginning of class)

SOLUTIONS (revised)

Question 1:

Let **A** and **B** be 4 by 4 matrices, and **c** be a length 4 column vector.

Note that to compute **ABc** it is faster to first multiply **Bc**.

Suppose, however, that we have a large number of 4-vectors and we want to multiply each on by **AB**. For a large enough number of vectors, it will be faster to compute **AB** and then multiply this intermediate result by each vector. How many vectors do we need to process in this way for it to be more efficient to compute this way?

Multiplying 2 4x4 matrices together is the same as doing 4 matrix/vector multiplies.

Doing the computation as A(Bc) does 2 matrix/vector multiplies, processing n vectors this way means doing 2n matrix vector multiplies.

First computing AB requires computing 4 matrix/vector multiplies, but then it only takes 1 matrix/vector multiply for each vector to process.

$4+n < 2*n$ when **n>4**

Question 2:

Consider the plane defined in 3 space 2x+2y+z=2.

What point on this plane is closest to the origin?

Does this plane intersect a unit sphere centered at the origin?

The shortest path between a point and a plane is in the direction of the normal to the plane. The normal vector of the plane is (2,2,1). A line starting at the origin and moving in this direction intersects the plane at **(4/9,4/9,2/9)** (one way to find this is to define the line as (2t, 2t, t), plug into the equation of the plane, and find t).

Since $(4/9)^2 + (4/9)^2 + (2/9)^2 < 1$, this point is inside the unit sphere, so **yes** the plane must intersect the sphere somewhere.

Question 3:

Is the space spanned by the following vectors a line, plane, or something larger?

3A: **a**=(1,2,3), **b**=(2,4,6), **c**=(3,6,9)

line (b = 2a, c=3a)

3B: **a**=(1,2,3), **b**=(-2,-4,-6), **c**=(3,6,9)

line (b=-2a, c=3a)

3C: **a**=(1,2,3), **b**=(2,-4,6), **c**=(3,6,9)

plane (since they aren't all on the same line)

3D: **a**=(4,0,3,-2), **b**==(-8,0,-6,4), **c**=(-2,0,-3/2,1)

line (b=-2a, c=-(1/2)a)

Question 4:

a and **b** are unit 3-vectors , Let **c**=**a**×**b**, and **d**=**a**×**c**

If the vectors **a, c,** and **d** do not form a basis for 3-space, what values can **a•b** have?

Under "normal" circumstances, c is perpendicular to a and b (by the definition of the cross product), so d would be perpindicular to a and c, do the three vectors form a basis for 3 space. The only way for this to fail is if a and c were not perpindicular, which would require either a or b to be 0 (which they can't be since we said they were unit vectors), or for a and b to be the same line, which means a•b = 1 or -1.

Question 5:

Prove that if there are 3 points on the plane (x1, y1), (x2,y2), (x3,y3), that the determinant of the 3 by 3 Matrix $\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix}$ has determinant 0 if the three points are co-linear.

The easiest way I know of to do this (and there are many others), is to say that the line is either vertical (in which case all the x are the same), or the line is y=mx+b (standard slope+intercept form).

Plugging both cases in...

Det($\begin{bmatrix} x & x & x \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix}$) = y1(x-x) + y2(x-x) + y3(x-x) = 0

Or...

Det($\begin{bmatrix} x_1 & x_2 & x_3 \\ mx1+b & mx2+b & mx3+b \\ 1 & 1 & 1 \end{bmatrix}$) =

x1 (mx2+b) + x2 (mx3+b) + x3(mx1+b) - x1(mx3+b) - x2(mx1+b) - x3(mx2+b)

a little term rearrangement...

b(x1+x2+x3-x1-x2-x3) + m(x1x2 + x2x3 + x3x1 - x1x3 - x2x1 - x3x2)

gives something that is obviously 0